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On Best Approximation by Rational and Holomorphic Mappings between Banach Spaces

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After proving a generalized version of Garkavi's theorem, we give as applications proofs of existence results on best approximation by polynomials, and fractional linear and holomorphic operators between Banach spaces. We also obtain theorems on best approximation by some types of rational functions defined in open subsets of Banach spaces. By considering a natural non-normable distance we prove that every mapping bounded on the bounded subsets of a Banach space has best approximation by polynomials of degree less than or equal to a fixed natural number n. © 1989 Academic Press, Inc.

1. INTRODUCTION

A classical result of Ch. de la Vallée-Poussin [13] states that every real continuous function on $[-1, 1]$ has a best approximation in the set of all functions of the form $P(x)/Q(x)$ where P and Q are real polynomials of degree less than or equal to m and n, respectively, and $Q(x) > 0$ for every $x \in [-1, 1]$. Walsh [14] proved a similar result for complex functions, continuous in a perfect subset of the complex plane. Cheney and Loeb [3] considered the problem of best approximation by ratios of trigonometric functions. Newman and Shapiro [10], Rice [11], and Bohem [2] studied the existence of best approximation by quotients of finite linear combinations of real continuous functions in topological spaces. Other aspects of the theory of best approximation by similar functions have been studied by many authors.

When U is a non-void open bounded subset of a complex Banach space E and F is also a complex Banach space, it makes sense to consider polynomials from U into F defined through continuous multilinear mappings from E into F. In this article we study the existence of best approximation of bounded mappings from U into F by certain quotients of polynomials from U into F by polynomials from U into $\mathbb{C}$ (i.e., rational mappings from

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U into F). The proof of our results depends on the compactness of certain subsets of holomorphic (i.e., Gâteaux-differentiable and continuous) mappings from U into F. We also prove results on best approximation of bounded mappings from U into F by holomorphic and polynomial mappings from U into F.

We denote by $l^\infty(U; F)$ the vector space of all bounded mappings from U into F with the norm

$$\|f\|_\infty = \sup\{\|f(x)\|; x \in U\}, \qquad f \in l^\infty(U; F).$$

The vector subspace of $l^\infty(U; F)$ formed by all bounded holomorphic mappings from U into F is denoted by $\mathscr{H}^\infty(U; F)$. We prove in Section 3 that, when F is a dual space, every $f \in l^\infty(U; F)$ has a best approximation in $\mathscr{H}^\infty(U; F)$ and, as a corollary to this result, that f has a best approximation in the set of all continuous polynomials from E into F with degree less than or equal to n.

A mapping $f \in \mathscr{H}^\infty(U; F)$ is called a rational mapping of type (m, n) if there are continuous polynomials P from E into F and Q from E into $\mathbb{C}$ of degrees less than or equal to m and n, respectively, such that $f(x)\,Q(x) = P(x)$ for every x in U and Q is not identically zero in U. We denote the set of all such mappings by $\mathscr{R}^\infty_{(m,n)}(U; F)$. In Section 4 we prove the existence of best approximations of $f \in l^\infty(U; F)$ by elements of $\mathscr{R}^\infty_{(m,n)}(U; F)$ when $\dim(E) < +\infty$ and F is a dual space. We also prove that when $\dim(E) = +\infty$ and F is $\mathbb{C}$ there exist best approximations of $f \in l^\infty(U; \mathbb{C})$ by elements of $\mathscr{R}^\infty_{(0,n)}(U; \mathbb{C})$ and $\mathscr{R}^\infty_{(1,1)}(U; \mathbb{C})$. The problem is open for the other values of m and n, but we conjecture that at least for the cases $m = 1$, $n \in \mathbb{N}$, $F = \mathbb{C}$, we should have results of existence on best approximation by rational functions of this type.

We denote by $\mathscr{F}_b(E; F)$ the vector space of all mappings from E into F which are bounded over the bounded subsets of E. The locally convex topology τ_b in $\mathscr{F}_b(E; F)$ of the uniform convergence over the bounded subsets of E is metrizable but non-normable in general. In Section 5 we prove results of best approximation of $f \in \mathscr{F}_b(E; F)$ by polynomial mappings from E into F with respect to a metric defining τ_b.

It is well known that the vector space of all compact linear mappings from E into F may be antiproximinal in the Banach space of all continuous linear mappings from E into F (see Holmes and Kripke [7]). However Deutsch *et al.* proved in [4] that, when F is a dual space, the set of continuous linear mappings from E into F of finite rank N (i.e., mappings whose images are contained in vector subspaces of dimension N) is proximinal in the Banach space of all bounded linear mappings from E into F. In Section 6, with the help of a result communicated to us by J. Mujica and a result of K. Floret [5], we show how theorems of this type are easily proved for holomorphic, rational, and polynomial mappings of finite rank N.

The lemma (and its corollary) proved in Section 2 is fundamental for the proofs of our results. It generalizes a result of Garkavi [6] and it is stated in greater generality than is necessary for our applications in greater generality than is necessary for our applications but, since it is interesting in itself, we felt we should state and prove it in this way.

2. The Fundamental Lemma

If X is a separated topological vector space over $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$) with topology τ we consider the set $\mathscr{S}(X)$ of all functions φ from X into $\mathbb{R}$ such that (i) $\varphi(x) \geqslant 0$ for every $x \in X$, (ii) φ is continuous in X, (iii) for every bounded subset B of X we have $\|\varphi\|_B \leqslant \operatorname{diam} \varphi$, where $\|\varphi\|_B = \sup\{\varphi(t); t \in B\}$ and $\operatorname{diam} \varphi = \sup\{\varphi(x); x \in X\} = \sup\{\varphi(y-x); y, x \in X\}$.

2.1. Examples. (a) If $p \in (0, 1]$ and g is a non-zero continuous p-seminorm in $(X; \tau)$ then $g \in \mathscr{S}(X)$ with $\operatorname{diam} g = +\infty$.

(b) If $p \in (0, 1]$ and the topology τ of X is defined by a sequence $(q_n)_{n=1}^{\infty}$ of p-seminorms in X, then we may consider

$$\varphi(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{q_n(x)}{1 + q_n(x)}, \qquad \forall x \in X$$

and

$$d(x, y) = \varphi(y - x), \qquad \forall x, y \in X.$$

Then d is a metric defining the topology τ of X. It is clear that φ is continuous in X and that $\operatorname{diam} \varphi = 1$. In order to show that $\varphi \in \mathscr{S}(X)$ it is enough to prove that for $n = 1, 2, \ldots$ and for every non-empty bounded subset B of (X, τ)

$$\sup\left\{\frac{q_n(x)}{1 + q_n(x)}; x \in B\right\} < 1 \tag{1}$$

which implies $\|\varphi\|_B < 1 = \operatorname{diam} \varphi$. If (1) were not true there would be a positive integer n such that for every $k = 1, 2, \ldots$ we could find $x_k \in B$ satisfying

$$\frac{q_n(x_k)}{1 + q_n(x_k)} > 1 - \frac{1}{k}.$$

This would give $q_n(x_k) > k-1$ for every $k = 1, 2, \ldots$. But this is impossible since q_n has to be bounded over B.

(c) If q is a non-zero continuous quasi-seminorm in (X, τ) then $q \in \mathscr{S}(X)$ with diam $q = +\infty$.

2.2. DEFINITION. If (X, τ) is a separated topological vector space over $\mathbb{K}$ and $\varphi \in \mathscr{S}(X)$, a non-empty subset Y of X is said to have the *Chebyshev center property in X relative to φ* if for every non-empty bounded subset B of X there is $f \in Y$ such that

$$\sup_{x \in B} \varphi(f-x) = \inf_{g \in Y} \sup_{x \in B} \varphi(g-x). \tag{2}$$

In this case f is called a *Chebyshev center of B relative to Y and φ*, and the right-hand side of (2) is called the *radius of Chebyshev of B relative to Y and φ*. If $B = \{x\}$ *we get* (2) writen in the form

$$\varphi(f-x) = \inf_{g \in Y} \varphi(g-x) \tag{3}$$

and we say that f is a *best approximation of x in Y relative to φ*. If this happens for all $x \in X$ we say that Y is *proximinal in X relative to φ*. When there is no doubr about the φ which is being considered we drop out the reference to φ (e.g., Y has the relative Chebyshev center property in X, f is a Chebyshev center of X relative to Y, etc).

When Y is proximinal in X relative to φ and $\varphi^{-1}(\{0\}) = \{0\}$, then it is quite simple to prove that Y is a closed subset of X for the topology τ.

2.3. LEMMA. *Let (X, τ) be a separated topological vector space over $\mathbb{K}$ and let φ be an element of $\mathscr{S}(X)$. If $\Delta_r = \{t \in \mathbb{R}; |t| \leq r\}$ we consider a separated topology σ in X compatible with the vector space structure such that $\varphi^{-1}(\Delta_r)$ is σ-closed for every $r \in [0, \text{diam } \varphi)$. We denote*

$$K_{r,\varphi}(B) = \{x \in X; x \in b + \varphi^{-1}(\Delta_r)\ \forall b \in B\}.$$

If Y is a non-empty subset of X such that $Y \cap K_{r,\varphi}(B)$ is σ-countably compact for every $r \in [0, \text{diam } \varphi)$ and every non-empty bounded subset B of (X, τ), then Y has the Chebyshev center property in X relative to φ.

Proof. For a non-empty bounded subset B of (X, τ) we consider

$$r_B = \inf_{y \in Y} \sup_{t \in B} \varphi(y-t) < \text{diam } \varphi$$

and we define $f_B(x)=\sup\{\varphi(x-t); t\in B\}$ for every x in X. For $\rho\in\mathbb{R}$ we have

$$\{x\in X; \varphi(x-t)\leqslant\rho\}=t+\varphi^{-1}(\Delta_\rho)$$

σ-closed for every $t\in B$. Hence f_B is σ-lower semicontinuous in X and, consequently, for $n=1, 2, \ldots$ and $\delta=\min\{1, \text{diam}\ \varphi-r_B\}$

$$C_n=\left\{y\in Y; f_B(y)\leqslant r_B+\frac{\delta}{n}\right\}\neq\phi$$

is relatively σ-closed in Y. We also have $C_{n+1}\subset C_n$ and $Y\cap K_{r_B+2^{-1}\delta,\varphi}(B)\supset C_n$ for every $n\geqslant 2$. Since $Y\cap K_{r_B+2^{-1}\delta,\varphi}(B)$ is σ-countably compact, it follows that $\bigcap_{n=2}^{\infty} C_n\neq\phi$. Hence we have $r_B=\sup\{\varphi(f-t); t\in B\}$ for each $f\in\bigcap_{n=2}^{\infty} C_n$. This means that each element of $\bigcap_{n=2}^{\infty} C_n$ is a Chebyshev center of B relative to Y and φ. Q.E.D.

2.4. Corollary. *Let X be a vector space over $\mathbb{K}$ and let q be either a p-norm ($p\in(0, 1]$) or a quasi-norm in X. If σ is a topology in X compatible with the vector space structure such that $\bar{B}_{q,1}(0)=\{x\in X; q(x)\leqslant 1\}$ is σ-closed and Y is a non-empty subset of X such that $\{y\in Y; q(y)\leqslant r\}$ is σ-countably compact for every $r>0$, then Y has the Chebyshev center property in X relative to q.*

Proof. First we note that $q^{-1}(\Delta_r)=\{x\in X; q(x)\leqslant r\}=\bar{B}_{q,r}(0)$ is the closed ball of center 0 and radius r with respect to q. If q is a p-norm $q^{-1}(\Delta_r)=r^{1/p}\bar{B}_{q,1}(0)$, and, if q is a quasi-norm $q^{-1}(\Delta_r)=r\bar{B}_{q,1}(0)$. In any case $q^{-1}(\Delta_r)$ is σ-closed. If B is a non-empty bounded subset of (X, q), then there is $\rho\geqslant 0$ such that $\sup\{q(t); t\in B\}\leqslant\rho$. If $r\geqslant 0$ and q is a p-norm we have

$$\begin{aligned} Y\cap K_{r,q}(B) &= \bigcap_{b\in B}\{y\in Y; q(y-b)\leqslant r\} \\ &\subset\{y\in Y; q(y)\leqslant r+\rho\}. \end{aligned}$$

If q is a quasi-norm, we know that there is $M\geqslant 0$ such that $q(z+t)\leqslant Mq(z)+Mq(t)$ for all z and t in X. Hence

$$Y\cap K_{r,q}(B)\subset\{y\in Y; q(y)\leqslant M(r+\rho)\}.$$

In any case we get $Y\cap K_{r,q}(B)$ σ-countably compact. Now we apply Lemma 2.3 with $\varphi=q$. Q.E.D.

3. Best Approximation by Holomorphic Operators

In this section we consider E a complex Banach space, U a non-void bounded open subset of E, and $F = G^*$ a dual Banach space. We denote by $l^\infty(U; F)$ the vector space of all bounded mappings from U into F normed by

$$\|f\|_\infty = \sup\{\|f(x)\|; x \in U\} \qquad \forall f \in l^\infty(U; F).$$

The Banach subspace of $l^\infty(U; F)$ formed by all bounded holomorphic (i.e., Gâteaux-differentiable and continuous) mappings from U into F will be denoted by $\mathscr{H}^\infty(U; F)$. The locally convex topology in $l^\infty(U; F)$ generated by the seminorms

$$P_{K,y}(f) = \sup\{|f(x)(y)|; x \in K\}$$

for $f \in l^\infty(U; F)$, K a compact subset of U, and $y \in G$, is denoted by τ_0^*. The compact-open topology in $l^\infty(U; F)$ is indicated by τ_0 and it is clear that $\tau_0 = \tau_0^*$ when F is a finite-dimensional Banach space.

3.1. Theorem. (1) *If $\mathscr{V}$ is a vector subspace of $l^\infty(U; F)$ containing $\mathscr{H}^\infty(U; F)$, then $\mathscr{H}^\infty(U; F)$ has the relative Chebyshev center property (hence, it is proximinal) in $\mathscr{V}$.*

(2) *If $\mathscr{W}$ is a τ_0^*-closed subset of $\mathscr{H}^\infty(U; F)$ and $\mathscr{V}$ is a vector subspace of $l^\infty(U; F)$ containing $\mathscr{W}$, then $\mathscr{W}$ has the relative Chebyshev center property in $\mathscr{V}$.*

Proof. (1) is a consequence of Corollary 2.4 if we prove that

$$\mathscr{B}_r = \{f \in \mathscr{H}^\infty(U; F): \|f\|_\infty \leqslant r\}$$

is τ_0^*-compact for every $r \geqslant 0$. By the generalized Montel's theorem (see Barroso *et al.* [1]) $\mathscr{B}_r$ is τ_0^*-relatively compact in $\mathscr{H}(U; (F, \sigma(F; G)))$. Here $\mathscr{H}(U; (F, \sigma(F; G)))$ denotes the vector space of all holomorphic (i.e., Gâteaux-differentiable and continuous) mappings from U into $(F, \sigma(F; G))$ and $\sigma(F; G)$ denotes the weak topology in F defined by G. If f is in the τ_0^*-closure of $\mathscr{B}_r$ in $\mathscr{H}(U; (F; \sigma(F, G)))$ there is a net $(f_\alpha)_{\alpha \in I}$ in $\mathscr{B}_r$ which is τ_0^*-convergent to f. It follows that $(|f_\alpha(x)(z)|)_{\alpha \in I}$ converges to $|f(x)(z)|$ for every $x \in U$ and $z \in G$. Hence

$$\|f\|_\infty = \sup_{\substack{z \in G, \|z\| \leqslant 1 \\ x \in U}} |f(x)(z)| \leqslant r$$

and $f \in l^\infty(U; F)$. Since $\mathscr{H}^\infty(U; F)$ is τ_0^*-closed in $l^\infty(U; F)$ it follows that $f \in \mathscr{B}_r$. Hence $\mathscr{B}_r$ is τ_0^*-compact.

Part (2) is a consequence of Corollary 2.4 since

$$\mathscr{W} \cap \mathscr{B}_r = \{f \in \mathscr{W};\ \|f\|_\infty \leqslant r\}$$

is τ_0^*-compact. Q.E.D.

Part (2) of this theorem gives results of best approximation by polynomial operators. In order to give the precise results of this type we fix the notation we are going to use. If $n = 1, 2, \ldots$ we consider the complex vector space $\mathscr{L}(^nE; F)$ of all continuous n-linear mappings from E^n into F. We denote by $\mathscr{P}(^nE; F)$ the vector space formed by all mappings P from E into F such that there is $A \in \mathscr{L}(^nE; F)$ satisfying $P(x) = A(x, \ldots, x) = Ax^n$ for all $x \in E$. For $n = 0$ the vector space $\mathscr{P}(^0E; F)$ is formed by all constant mappings from E into F. The elements of $\mathscr{P}(^nE; F)$, $n = 0, 1, \ldots$, are called *n-homogeneous continuous polynomials from E into F*. If we set

$$\|P\| = \sup\{\|P(x)\|;\ \|x\| \leqslant 1\} \qquad \forall P \in \mathscr{P}(^nE; F)$$

then $\mathscr{P}(^nE; F)$ is a Banach space and it is not difficult to show that $\|\cdot\|_\infty$ is an equivalent norm in this space. Hence we may consider $\mathscr{P}(^nE; F)$ as a Banach subspace of $l^\infty(U; F)$ through the restriction mapping to U. A mapping $P\colon E \to F$ is called a *continuous polynomial of degree less than or equal to* $m \in \mathbb{N} = \{0, 1, \ldots\}$ if $P = P_0 + P_1 + \cdots + P_m$ for some $P_j \in \mathscr{P}(^jE; F)$, $j = 0, 1, \ldots, m$. The vector space of all such mappings will be denoted by $\mathscr{P}_m(E; F)$. For all $n, m \in \mathbb{N}$ the subspaces $\mathscr{P}(^nE; F)$ and $\mathscr{P}_m(E; F)$ are τ_0^*-closed in $\mathscr{H}^\infty(U; F)$. Hence, from Theorem 3.1, part (2), it follows that the following results are true for all $n, m \in \mathbb{N}$.

3.2. COROLLARY (1) *The vector space $\mathscr{P}_m(E; F)$ of all continuous polynomials from E into F of degree less than or equal to m has the relative Chebyshev center property (hence, it is proximinal) in $l^\infty(U; F)$.*

(2) *The vector space $\mathscr{P}(^nE; F)$ of all continuous n-homogeneous polynomials from E into F has the relative Chebyshev center property (hence, it is proximinal) in $l^\infty(U; F)$.*

The special case of part (2) in Corollary 3.2 was proved by Roversi in [12].

4. BEST APPROXIMATION BY RATIONAL MAPPINGS

In this section E is a complex Banach space, U is a non-empty bounded open subset of E, and F is a complex dual Banach space.

We denote by $\mathscr{R}^\infty_{(m,n)}(U; F)$ the set of all $f \in \mathscr{H}^\infty(U; F)$ such that there are polynomials $P \in \mathscr{P}_m(E; F)$, $Q \in \mathscr{P}_n(E; \mathbb{C})$ satisfying $Q(x) f(x) = P(x)$ for every $x \in U$ with Q not identically zero in U. The elements of $\mathscr{R}^{\infty}_{(m,n)}(U; F)$ are called *bounded rational mappings of type* (m, n) *from* U *into* F. We note that:

(i) The elements of $\mathscr{R}^{\infty}_{(m,0)}(U; F)$ are the restrictions to U of the polynomials of degree less than or equal to m.

(ii) If $f \in \mathscr{R}^{\infty}_{(0,n)}(U; F)$ there are $c \in F$, $Q \in \mathscr{P}_n(E; \mathbb{C})$ such that $Q(x) f(x) = c$ for every $x \in U$ and Q is not identically zero in U. If f is not the constant mapping 0 we have $Q \cdot f$ not identically zero in an open dense subset of U. Hence $c \neq 0$ and it follows that $f(x) \neq 0$ and $Q(x) \neq 0$ for every $x \in U$. Therefore $f(x) = c/Q(x)$ for every $x \in U$.

(iii) The elements of $\mathscr{R}^{\infty}_{(1,1)}(U; F)$ are called bounded linear fractional mappings from U into F.

The next lemma is fundamental in the proof of the results we get on best approximation by rational mappings.

4.1. Lemma. *Let $\mathscr{B}_r$ be the subset of $\mathscr{R}^{\infty}_{(m,n)}(U; F)$ formed by those mappings f such that $\|f\|_{\infty} \leqslant r$. If $(f_j)_{j=1}^{\infty}$ is a sequence of elements of $\mathscr{B}_r$, then there are $f \in \mathscr{H}^{\infty}(U; F)$, $x_0 \in U$, and a subnet $(f_{j_\alpha})_{\alpha \in I}$ of $(f_j)_{j=1}^{\infty}$ such that $(f_{j_\alpha})_{\alpha \in I}$ converges to f in the sense of the topology τ_0^* and, for every finite-dimensional vector subspace S of E with $x_0 \in S$, we have $f \mid U \cap S$ as an element of $\mathscr{R}^{\infty}_{(m,n)}(U \cap S; F)$.*

Proof. For every $j = 1, 2, \ldots$ there are $P_j \in \mathscr{P}_m(E; F)$, $Q_j \in \mathscr{P}_n(E; \mathbb{C})$ such that $Q_j(x) f_j(x) = P_j(x)$ for every $x \in U$ and Q_j is not identically zero in U. With no loss of generality we may take $\|Q_j\|_{\infty} = 1$ for every $j = 1, 2, \ldots$. Hence $\|P_j\|_{\infty} \leqslant r$ for every $j = 1, 2, \ldots$. By the generalized version of Montel's theorem we can get $f \in \mathscr{H}^{\infty}(U; F)$, $P \in \mathscr{P}_m(E; F)$, $Q \in \mathscr{P}_n(E; \mathbb{C})$, and a subnet $(f_{j_\alpha})_{\alpha \in I}$ of $(f_j)_{j=1}^{\infty}$ such that $(f_{j_\alpha})_{\alpha \in I}$, $(P_{j_\alpha})_{\alpha \in I}$, and $(Q_{j_\alpha})_{\alpha \in I}$ converge respectively to f, P, and Q in the sense of the topology τ_0^*. It is clear that $f(x) Q(x) = P(x)$ for every $x \in U$. If f is identically zero in U the lemma is already proved. If f is not identically zero we consider the sets $A_j = \{x \in U; Q_j(x) \neq 0\}$, $j = 1, 2, \ldots$, and $A = \{x \in U; f(x) \neq 0\}$. These sets are open dense subsets of U. Hence, by Baire's theorem, $B = A \cap (\bigcap_{j=1}^{\infty} A_j)$ is dense in U and there is $x_0 \in U$ such that $f(x_0) \neq 0$ and $Q_j(x_0) \neq 0$ for every $j = 1, 2, \ldots$. If S is a finite-dimensional vector subspace of E with $x_0 \in S$, then $U \cap S$ is relatively compact in S and we have $\|Q_j\|_{U \cap S} = \sup\{\|Q_j(x)\|; x \in U \cap S\} = \sup\{\|Q_j(x)\|; x \in \overline{U \cap S}\} = \|Q_j\|_{\overline{U \cap S}}$. Since $(Q_{j_\alpha})_{\alpha \in I}$ converges to Q for τ_0^* in E we have $(\|Q_{j_\alpha}\|_{\overline{U \cap S}})_{\alpha \in I}$ converging to $\|Q\|_{\overline{U \cap S}}$. By dividing Q_j and P_j by $\|Q_j\|_{\overline{U \cap S}}$ we may consider $\|Q_j\|_{\overline{U \cap S}} = 1$ for every $j = 1, 2, \ldots$. It follows that $\|Q\|_{\overline{U \cap S}} = 1$ and Q is not identically zero in $U \cap S$. Hence $f \mid U \cap S$ is an element of $\mathscr{R}^{\infty}_{(m,n)}(U \cap S; F)$.

Q.E.D.

We remark that the above proof does not provide us with Q not identically zero in U since we modified the Q_j's when we divided them by $\|Q_j\|_{\overline{S \cap U}}$. Now we can prove the following results.

4.2. THEOREM. *When* $\dim(E) < \infty$

(i) *If* $\mathscr{V}$ *is a vector subspace of* $l^\infty(U; F)$ *containing* $\mathscr{R}_{(m,n)}(U; F)$, *then* $\mathscr{R}^\infty_{(m,n)}(U; F)$ *has the relative Chebyshev center property* (*hence, it is proximinal*) *in* $\mathscr{V}$.

(ii) *If* $\mathscr{W}$ *is a non-empty* τ_0^**-closed subset of* $\mathscr{R}^\infty_{(m,n)}(U; F)$ *and* $\mathscr{V}$ *is a vector subspace of* $l^\infty(U; F)$ *containing* $\mathscr{W}$, *then* $\mathscr{W}$ *has the relative Chebyshev center property in* $\mathscr{V}$.

4.3. THEOREM. (i) *If* $\mathscr{V}$ *is a vector subspace of* $l^\infty(U; \mathbb{C})$ *containing* $\mathscr{R}^\infty_{(0,n)}(U; \mathbb{C})$ (*respectively,* $\mathscr{R}^\infty_{(1,1)}(U; \mathbb{C})$), *then* $\mathscr{R}^\infty_{(0,n)}(U; \mathbb{C})$ (*respectively,* $\mathscr{R}^\infty_{(1,1)}(U; \mathbb{C})$) *has the relative Chebyshev center property in* $\mathscr{V}$.

(ii) *If* $\mathscr{W}$ *is a* τ_0*-closed non-empty subset of either* $\mathscr{R}^\infty_{(0,n)}(U; \mathbb{C})$ *or* $\mathscr{R}^\infty_{(1,1)}(U; \mathbb{C})$ *and* $\mathscr{V}$ *is a vector-subspace of* $l^\infty(U; \mathbb{C})$ *containing* $\mathscr{W}$, *then* $\mathscr{W}$ *has the relative Chebyshev center property in* $\mathscr{V}$.

Proof of Theorem 4.2. (i) follows from Corollary 2.4 since, when E has finite dimension, Lemma 4.1 implies that $\mathscr{B}_r = \{f \in \mathscr{R}^\infty_{(m,n)}(U; F); \|f\|_\infty \leqslant r\}$ is τ_0^*-countably compact. Part (ii) follows from the fact that $\mathscr{B}_r \cap \mathscr{W} = \{f \in \mathscr{W}; \|f\|_\infty \leqslant r\}$ is τ_0^*-countably compact.

Proof of Theorem 4.3. Part (i) will be proved as a consequence of Corollary 2.4 if we show that $\mathscr{B}_r = \{f \in \mathscr{R}^\infty_{(m,n)}(U, \mathbb{C}); \|f\|_\infty \leqslant r\}$ is τ_0-countably compact when (a) $m = 0$ and (b) $m = n = 1$.

Case (a). Let $(f_j)_{j=1}^\infty$ be a sequence in $\mathscr{B}_r$. By Lemma 4.1 we know that there are $f \in \mathscr{H}^\infty(U; \mathbb{C})$, $x_0 \in U$, a subset $(f_{j_\alpha})_{\alpha \in I}$ of $(f_j)_{j=1}^\infty$ such that $(f_{j_\alpha})_{\alpha \in I}$ converges to f in the sense of the topology τ_0, and, for every finite-dimensional vector subspace S of E with $x_0 \in S$, we have $f \mid U \cap S \in \mathscr{R}^\infty_{(0,n)}(U \cap S; \mathbb{C})$. Since $\|f\|_\infty \leqslant r$ it is enough to show that $f \in \mathscr{R}^\infty_{(0,n)}(U; \mathbb{C})$. If $f = 0$ this is trivial. We suppose $f \neq 0$. For each above-mentioned S we can find $c_S \in \mathscr{P}_0(S; \mathbb{C}) = \mathbb{C}$ and $Q^S \in \mathscr{P}_n(S; \mathbb{C})$ such that $f(x) = c_S / Q^S(x)$ for every $x \in U \cap S$ and $Q^S(x) \neq 0$ for every $x \in U \cap S$. (See the remark about $\mathscr{R}^\infty_{(0,n)}(U; F)$ made at the beginning of this section.) By examining the proof of Lemma 4.1 it is clear we may consider $f(x_0) \neq 0$. Hence $c_S \neq 0$ for every S and we may consider $c_S = 1$ for every S. We consider the Taylor series developments of Q_S and f aroud x_0 and we write

$$Q^S = \sum_{j=0}^{n} Q_j^S$$

$$f = \sum_{j=0}^{\infty} f_j,$$

where $f_j \in \mathscr{P}({}^jE; \mathbb{C})$ for $j \in \mathbb{N}$, $Q_j^S \in \mathscr{P}({}^jS; \mathbb{C})$ for $j = 0, 1, ..., n$, and the equality holds true in a neighborhood of x_0 in $U \cap S$. Since $f \cdot Q^S = 1$ in $U \cap S$ the unicity of the Taylor series development implies that in S we have

$$f_0 Q_0^S = 1$$

$$f_1 Q_0^S + f_0 Q_1^S = 0$$

$$\dots\dots\dots\dots\dots\dots$$

$$f_n Q_0^S + f_{n-1} Q_1^S + \cdots + f_0 Q_n^S = 0$$

$$f_k Q_0^S + f_{k-1} Q_1^S + \cdots + f_{k-n} Q_n^S = 0, \qquad \text{for} \quad k \geqslant n+1.$$

Hence, since $f_0 = f(x_0) \neq 0$, we have $f_0^{n+1} \neq 0$ and the first $n+1$ above equations have a unique solution $Q_0^S, ..., Q_n^S$ defined in S by expressions in terms of $1, f_0, ..., f_n$ (by the so-called Cramer's rule). If we define $Q_0, ..., Q_n$ in E by the same expressions (it makes sense to do it because $f_1, ..., f_n$ are defined in E and $1, f_0$ are non-zero constants) we get $Q = Q_0 + \cdots + Q_n \in \mathscr{P}_n(E; \mathbb{C})$ satisfying Q not identically zero in U and $Q \cdot f = 1$ in U (since $Q \mid S \cdot f = Q^S \cdot f = 1$ in $U \cap S$ for every S). Thus f is in $\mathscr{R}^{\infty}_{(0,n)}(U; \mathbb{C})$.

Case (b). Let $(f_j)_{j=1}^{\infty}$ be a sequence in $\mathscr{B}_r$. By Lemma 4.1 we know that there are $f \in \mathscr{H}^{\infty}(U; \mathbb{C})$, $x_0 \in U$, a subnet $(f_{j_\alpha})_{\alpha \in I}$ of $(f_j)_{j=1}^{\infty}$ such that $(f_{j_\alpha})_{\alpha \in I}$ converges to f in the sense of the topology τ_0, and, for every finite-dimensional vector subspace S of E with $x_0 \in S$, we have $f \mid S \cap U$ in $\mathscr{R}^{\infty}_{(1,1)}(U; \mathbb{C})$. Since $\|f\|_{\infty} \leqslant r$ it is enough to show that $f \in \mathscr{R}^{\infty}_{(1,1)}(U; \mathbb{C})$. This is trivial if f is identically zero in U. We suppose that this is not the case. For every S we consider $P^S, Q^S \in \mathscr{P}_1(S; \mathbb{C})$ such that $f(x) \cdot Q^S(x) = P^S(x)$ for each $x \in S \cap U$ and Q^S not identically zero in $U \cap S$. Now we consider the Taylor series developments of P^S, Q^S, and f in a neighborhood of x_0 in $S \cap U$:

$$P^S = P_0^S + P_1^S, \qquad Q^S = Q_0^S + Q_1^S, \qquad f = \sum_{j=0}^{\infty} f_j.$$

Here $P_0^S, Q_0^S \in \mathbb{C}$, $P_1^S, Q_1^S \in \mathscr{P}({}^1S; \mathbb{C})$, $f_j \in \mathscr{P}({}^jE; \mathbb{C})$, $j \in \mathbb{N}$. By the unicity of the Taylor series development the equality $P^S = Q^S \cdot f$ in a neighborhood of x_0 in $U \cap S$ implies the following equalities in S:

$$f_0 Q_0^S = P_0^S$$
$$f_1 Q_0^S + f_0 Q_1^S = P_1^S$$
$$f_j Q_0^S + f_{j-1} Q_1^S = 0 \qquad \text{for} \quad j \geqslant 2.$$

As we saw in the proof of Lemma 4.1 we can always consider $f(x_0) = f_0 \neq 0$. We have two possibilities to consider:

(1) For every finite-dimensional vector subspace S of E such that $x_0 \in S$ there is another such vector subspace $S' \supset S$ satisfying $P_0^{S'} = 0$.

(2) There is a finite-dimensional vector subspace S_0 of E such that $x_0 \in S_0$ and for every other such subspace S of E, $S \supset S_0$ we have $P_0^S \neq 0$.

In case (1), if $P_0^{S'} = 0$ it follows that $Q_0^{S'} = 0$ since $f_0 \neq 0$. Then $f_0 Q_1^{S'} = P_1^{S'}$ and $f_{j-1} Q_1^{S'} = 0$ for $j \geqslant 2$. Since $Q^{S'} \neq 0$ and $Q_0^{S'} = 0$, we must have $Q_1^{S'} \neq 0$ in an open dense subset of $U \cap S'$. Thus f_{j-1} is identically zero in this set and $f_{j-1} | S' = 0$ for $j \geqslant 2$. Therefore f is constant in $S' \cap U$. But under our hypothesis of case (1) it follows that f is constant in U and hence $f \in \mathscr{R}^{\infty}_{(0,0)}(U; \mathbb{C}) \subset \mathscr{R}^{\infty}_{(1,1)}(U; \mathbb{C})$.

In case (2) with no loss of generality we may suppose that $P_0^S = 1$ for every finite-dimensional vector subspace S of E containing S_0. It follows that $Q_0^S = 1/f_0$ and $Q_1^S = P_1^S/f_0 - f_1/f_0^2$ in S. If we replace these values in the equations $f_j Q_0^S + f_{j-1} Q_1^S = 0$ for $j \geqslant 2$ we get in S

$$\frac{f_{j-1}}{f_0} P_1^S = \frac{f_{j-1} \cdot f_1}{f_0^2} - \frac{f_j}{f_0}.$$

If for some $j \geqslant 2$, $f_{j-1} \neq 0$ in E we have $f_{j-1}(x) \neq 0$ for every x in an open dense subset V of E. Hence

$$P_1^S(x) = \frac{f_1(x)}{f_0} - \frac{f_j(x)}{f_{j-1}(x)}$$

for every $x \in S \cap V$. For all those S such that $S \cap V \neq \phi$ the right-hand side of the above equation defines a continuous function in an open dense subset $V \cap S$ of $U \cap S$ and (by the left-hand side) it has a continuous linear extension to S equal to P_1^S. Since the right-hand side is independent of the S we consider, by defining

$$P_1(x) = \frac{f_1(x)}{f_0} - \frac{f_j(x)}{f_{j-1}(x)}$$

for $x \in V$ we get a continuous function in V which has a linear extension $P_1 \in \mathscr{P}(^1E; \mathbb{C})$. We may take

$$Q_1 = \frac{P_1}{f_0} - \frac{f_1}{f_0^2} \in \mathscr{P}(^1E; \mathbb{C})$$

and we get $Q_1 \mid S = Q_1^S$. Hence $f \cdot (Q_1 + Q_0) = P_1 + P_0$ where $Q_0 = 1/f_0$ and $P_0 = 1$ in U with $Q_0 + Q_1 \neq 0$. Hence $f \in \mathscr{R}^{\infty}_{(1,1)}(U; \mathbb{C})$.

If for any $j \geqslant 2$ we have $f_{j-1} = 0$ in E, then f is constant in U and it belongs to $\mathscr{R}^{\infty}_{(0,0)}(U; \mathbb{C}) \subset \mathscr{R}^{\infty}_{(1,1)}(U; \mathbb{C})$. Q.E.D.

The question of density of rational functions in the set of holomorphic functions over compact subsets of Banach spaces was examined by Matos in [8].

5. Best Non-normable Metric Approximation by Polynomial Operators

In this section E, F, and G are complex Banach spaces and $F = G^*$. We denote $\mathscr{F}_b(E; F)$ the complex vector space of all mappings from E into F which are bounded over the bounded subsets of E. The set of all bounded subsets of E is indicated by $b(E)$. If $B \in b(E)$ and $f \in \mathscr{F}_b(E; F)$ we set

$$\|f\|_B = \sup \{\|f(t)\|; t \in B\}.$$

The locally convex topology τ_b in $\mathscr{F}_b(E; F)$ generated by the family of seminorms $(\|\cdot\|_B)_{B \in b(E)}$ is metrizable. A corresponding metric defining this topology is given by

$$|f - g| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{B_n}}{1 + \|f - g\|_{B_n}},$$

where $(B_n)_{n=1}^{\infty}$ is an increasing sequence of elements of $b(E)$ such that $E = \bigcup_{n=1}^{\infty} B_n$ and every $B \in b(E)$ is contained in some B_n (e.g., $B_n = \{x \in E; \|x\| \leqslant n\}$, 1, 2, ...). It is obvious that this metric depends on the sequence $(B_n)_{n=1}^{\infty}$ we take, but it is quite simple to see that all the results we are going to prove will be true for any one of these metrics. In order to simplify our notation we choose $B_n = \{x \in E; \|x\| \leqslant n\}$, $n = 1, 2, \ldots$. As it was shown in Example 2.1(b) (with $p = 1$) the function $\varphi(f) = |f| = |f - 0|$ for $f \in \mathscr{F}_b(E; F)$ is an element of $\mathscr{S}((\mathscr{F}_b(E; F), \tau_b))$ with diam $|\cdot| = 1$. We denote by τ_0^* the locally convex topology in $\mathscr{F}_b(E; F)$ generated by the seminorms $P_{K,z}$, where K is a compact subset of E and $z \in G$ (see Sect. 3 where we first considered $p_{K,z}$). Hence $\tau_0^* \subset \tau_b$. It is clear that the topology

ω^* in $\mathscr{F}_b(E; F)$ generated by the seminorms $p_{\{x\},z}$ with $x \in E$ and $z \in G$ is such that $\omega^* \subset \tau_0^*$.

5.1. LEMMA. *For $r \in [0, 1)$ the set*

$$\mathscr{D}_r = \{f \in \mathscr{F}_b(E; F); |f| \leqslant r\}$$

is ω^*-closed, hence τ_0^*-closed in $\mathscr{F}_b(E; F)$.

Proof. First we suppose that there is f in the ω^*-closure of $\mathscr{D}_r$ not belonging to $\mathscr{D}_r$. Hence $|f| > r$ and there is a net $(f_\alpha)_{\alpha \in A}$ in $\mathscr{D}_r$ converging to f for the ω^* topology. We consider $\rho = 2^{-1}(|f| - r) > 0$ and $k \in \mathbb{N}$ such that

$$\sum_{n=1}^{k} 2^{-n} \frac{\|f\|_{B_n}}{1 + \|f\|_{B_n}} > r + \rho. \tag{4}$$

If $n \in \{1, ..., k\}$ and $\delta_n > 0$, since

$$\|f\|_{B_n} = \sup\{|f(x)(t)|; x \in B_n, t \in G, \|t\| \leqslant 1\}$$

and

$$t \in \mathbb{R}^+ \to \frac{t}{1+t} \in \mathbb{R}^+$$

is continuous and increasing, there are $x_n \in B_n$, $t_n \in G$, $\|t_n\| \leqslant 1$ such that

$$2^{-n} \frac{\|f\|_{B_n}}{1 + \|f\|_{B_n}} - 2^{-n} \frac{|f(x_n)(t_n)|}{1 + |f(x_n)(t_n)|} < \delta_n. \tag{5}$$

Since $\lim_{\alpha \in A} |f_\alpha(x_n)(t_n)| = |f(x_n)(t_n)|$, for a given $\rho_n > 0$ there is $\alpha_n \in A$ such that $\alpha \in A$, $\alpha \geqslant \alpha_n$ implies

$$2^{-n} \frac{|f(x_n)(t_n)|}{1 + |f(x_n)(t_n)|} - 2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} < \rho_n. \tag{6}$$

Hence, for $\alpha \geqslant \alpha_n$, it follows from (5) and (6) that

$$2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} > 2^{-n} \frac{\|f\|_{B_n}}{1 + \|f\|_{B_n}} - (\rho_n + \delta_n). \tag{7}$$

Now if we consider δ_n and ρ_n such that

$$\sum_{n=1}^{k} (\delta_n + \rho_n) < \rho$$

and $\alpha_0 \in A$ such that $\alpha_0 \geqslant \alpha_n$ for $n = 1, ..., k$, it follows from (7) and (4) that

$$\sum_{n=1}^{k} 2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} > r$$

for every $\alpha \geqslant \alpha_0$. Thus

$$|f_\alpha| \geqslant \sum_{n=1}^{k} 2^{-n} \frac{\|f_\alpha\|_{B_n}}{1 + \|f_\alpha\|_{B_n}} \geqslant \sum_{n=1}^{k} 2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} > r$$

for all $\alpha \geqslant \alpha_0$. But this is impossible since $f_\alpha \in \mathscr{D}_r$ for every $\alpha \in A$. Hence we must have $f \in \mathscr{D}_r$. Q.E.D.

5.2. THEOREM. (a) *If* $\mathscr{U}$ *is a vector subspace of* $\mathscr{F}_b(E; F)$ *containing* $\mathscr{P}_m(E; F)$, *then* $\mathscr{P}_m(E; F)$ *has the Chebyshev center property* (*hence, it is proximinal*) *in* $\mathscr{U}$ *relative to* $|\cdot|$.

(b) *If* $\mathscr{W}$ *is a* τ_0^**-closed subset of* $\mathscr{P}_m(E; F)$ *and* $\mathscr{U}$ *is a vector subspace of* $\mathscr{F}_b(E; F)$ *containing* $\mathscr{W}$, *then* $\mathscr{W}$ *has the Chebyshev center property* (*hence, it is proximinal*) *in* $\mathscr{U}$ *relative to* $|\cdot|$.

Proof. Part (a) will follow from Lemma 2.3 and Lemma 5.1 if we show that for each τ_b-bounded subset $\mathscr{B}$ of $\mathscr{F}_b(E; F)$ and each $r \in [0, 1)$, the set

$$K_r(\mathscr{B}) = \bigcap_{Q \in \mathscr{B}} \{P \in \mathscr{P}_m(E; F); P \in Q + \mathscr{D}_r\}$$

is τ_0^*-compact. Let $k > 0$ be such that $k/(1+k) > r$. If $P \in \mathscr{P}_m(E; F)$ and $Q \in \mathscr{B}$ are such that $\|P - Q\|_{B_j} \geqslant k$ for all $j = 1, 2, ...$, then

$$|P - Q| \geqslant \sum_{j=1}^{\infty} 2^{-j} \frac{k}{1+k} = \frac{k}{1+k} > r$$

and $P \notin Q + \mathscr{D}_r$. Hence, if $P \in K_r(\mathscr{B})$ and $Q \in \mathscr{B}$, there is $j \in \{1, 2, ...\}$ such that

$$\|P - Q\|_{B_1} \leqslant \|P - Q\|_{B_j} \leqslant k.$$

It follows that

$$\|P\|_{B_1} \leqslant k + \|Q\|_{B_1} \leqslant k + \sup_{Q \in \mathscr{B}} \|Q\|_{B_1} = k + C < +\infty$$

and

$$\sup_{P \in K_r(\mathscr{B})} \|P\|_{B_1} \leqslant k + C < +\infty.$$

Thus $K_r(\mathscr{B})$ is contained in the closed ball of center 0 and radius $k + C$ in $\mathscr{P}_m(E; F)$ with respect to the norm $\|\cdot\|_{B_1}$. We denote this ball by $\mathscr{D}$. We know that for every $P \in \mathscr{P}_m(E; F)$

$$P = \sum_{j=0}^{m} P_j \qquad (P_j \in \mathscr{P}({}^jE; F),\ j = 0, ..., m)$$

where

$$P_j(x) = \frac{1}{2\pi i} \int_{|\lambda| = 1} \frac{P(\lambda x)}{\lambda^{j+1}}\, d\lambda$$

for every $x \in B_1$ (see Nachbin [9]). It follows that

$$\|P_j\|_{B_1} \leqslant \|P\|_{B_1} \leqslant k + C$$

for all $P \in \mathscr{D}$ and $j = 0, ..., m$. Hence

$$\sup_{P \in \mathscr{D}} \|P(x)\| \leqslant (k + C) \sum_{j=0}^{m} \|x\|^j < +\infty$$

and $\mathscr{D}(x) = \{P(x); P \in \mathscr{D}\}$ is $\sigma(F, G)$-relatively compact for every $x \in E$. If K is a compact subset of E and $z \in G$ we have

$$\sup_{P \in \mathscr{D}} p_{K,z}(P) \leqslant (k + C)\, \|z\| \sum_{j=0}^{m} \sup_{t \in K} \|t\|^j < +\infty.$$

Hence $\mathscr{D}$ is τ_0^*-bounded. By the generalized Montel's theorem $\mathscr{D}$ is τ_0^*-relatively compact in $\mathscr{H}(E; (F, \sigma(F, G)))$. In order to prove that $\mathscr{D}$ is τ_0^*-compact it is enough to show that $\mathscr{D}$ is τ_0^*-closed in $\mathscr{H}(E; (F, \sigma(F, G)))$. Let $(P_\alpha)_{\alpha \in I}$ be a net in $\mathscr{D}$ τ_0^*-convergent to $f \in \mathscr{H}(E; (F, \sigma(F, G)))$. We have $\lim_{\alpha \in I} P_\alpha(x)(z) = f(x)(z)$ for every $x \in E$ and $z \in G$. This implies that f is of the form

$$f(x) = \sum_{j=0}^{m} Q_j(x) \qquad \forall x \in E$$

with $Q_j(x) = A_j(x, ..., x) = A_j x^j$, $\forall x \in E$, where A_j is a j-linear mapping from E^j into F. Hence f is a polynomial (not necessarily continuous) of degree less than or equal to m. Since $|P_\alpha(x)(z)| \leqslant k + C$ for $x \in B_1$, $z \in G$, $\|z\| = 1$, we get $|f(x)(z)| \leqslant k + C$ for $x \in B_1$, $z \in G$, $\|z\| = 1$. This means that

$\sup_{x\in B_1}\|f(x)\|\leqslant k+C$. But we know that a polynomial bounded over the unit ball is continuous. Therefore $f\in\mathscr{P}_m(E;F)$ and $f\in\mathscr{Q}$. Since $K_r(\mathscr{B})$ is contained in the τ_0^*-compact subset $\mathscr{Q}$ of $\mathscr{P}_m(E;F)$ it is enough to prove that $K_r(\mathscr{B})$ is τ_0^*-closed in $\mathscr{P}_m(E;F)$ in order to show that $K_r(\mathscr{B})$ is τ_0^*-compact. If $P\in\mathscr{P}_m(E;F)$ is the τ_0^*-limit of a net $(P_\alpha)_{\alpha\in I}$ of $K_r(\mathscr{B})$, we have $P_\alpha-Q\in\mathscr{D}_r$ for every $\alpha\in I$ and $Q\in\mathscr{B}$. Since $\mathscr{D}_r$ is τ_0^*-closed in $\mathscr{F}_b(E;F)$ by Lemma 5.1, it follows that $P-Q=\lim_{\alpha\in I}P_\alpha-Q\in\mathscr{D}_r$. Hence $P\in K_r(\mathscr{B})$ as was our objective.

Part (b) follows from Lemma 2.3, Lemma 5.1, and from the fact that for every $\mathscr{B}$ τ_b-bounded in $\mathscr{F}_b(E;F)$ and every $r\in[0,1)$, the set

$$\bigcap_{Q\subset\mathscr{B}}\{P\in\mathscr{W};P\in Q+\mathscr{D}_r\}=\mathscr{W}\cap K_r(\mathscr{B})$$

is τ_0^*-compact. Q.E.D.

We note that for every $n\leqslant m$ the vector subspace $\mathscr{P}(^nE;F)$ is τ_0^*-closed in $\mathscr{P}_m(E;F)$. Hence part (b) of Theorem 5.2 implies that $\mathscr{P}(^nE;F)$ is proximinal in $\mathscr{F}_b(E;F)$ with respect to $|\cdot|$.

With the methods of this paper we cannot prove results of best approximation by holomorphic or rational mappings relative to $|\cdot|$. The problem is that, in general, the set of holomorphic mappings, corresponding to $K_{r,|\cdot|}(B)$ of Lemma 2.3, is not τ_0^*-compact.

6. Best Approximation by Finite Rank Operators

As we have considered before E, F, and G are complex Banach spaces with $F=G^*$ and U is a non-empty bounded open subset of E. In $l^\infty(U;F)$ and in $\mathscr{F}_b(E;F)$ we consider their subsets $l_N^\infty(U;F)$ and $\mathscr{F}_{b,N}(E;F)$ of all mappings whose images are contained in vector subspaces of F with finite dimension $\leqslant N$. Then we consider

$$\mathscr{H}_N^\infty(U;F)=\mathscr{H}^\infty(U;F)\cap l_N^\infty(U;F)$$
$$\mathscr{P}^N(^nE;F)=\mathscr{P}(^nE;F)\cap l_N^\infty(U;F)$$
$$\mathscr{P}_m^N(E;F)=\mathscr{P}_m(E;F)\cap l_N^\infty(U;F)$$
$$\mathscr{R}_{(m,n),N}^\infty(U;F)=\mathscr{R}_{(m,n)}^\infty(U;F)\cap l_N^\infty(U;F).$$

We recall the following results

6.1. Theorem. *$\mathscr{P}_N(^1E;F)=\mathscr{L}_N(E;F)$ is ω^*-closed in $\mathscr{P}(^1E;F)=\mathscr{L}(E;F)$.*

This result is due to K. Floret (see [5]).

6.2. THEOREM. *For an open subset U of E there are complex Banach spaces H_U^∞ and $\varepsilon_U \in \mathscr{H}^\infty(U; H_U^\infty)$ with the following universal property: for every complex Banach space H and every $f \in \mathscr{H}^\infty(U; H)$ there is a unique $T_f \in \mathscr{P}(^1H_U^\infty; H) = \mathscr{L}(H_U^\infty; H)$ such that $T_f \circ \varepsilon_U = f$, and $\|T_f\| = \|f\|$.*

This result has been communicated to us by Jorge Mujica and it will be published later.

It is easy to prove the following corollary to these two theorems:

6.3. COROLLARY. *$\mathscr{P}^N(^nE; F)$, $\mathscr{P}_m^N(E; F)$, $\mathscr{R}_{(m,n),N}^\infty(U; F)$, and $\mathscr{H}_N^\infty(U; F)$ are τ_0^*-closed in $\mathscr{P}(^nE; F)$, $\mathscr{P}_m(E; F)$, $\mathscr{R}_{(m,n)}^\infty(U; F)$, $\mathscr{H}^\infty(U; F)$, respectively.*

With this corollary and Theorems 3.1, 4.1, 4.2, 5.2 we get immediately the following results.

6.4. THEOREM. (i) *$\mathscr{P}^N(^nE; F)$ and $\mathscr{P}_m^N(E; F)$ have the relative Chebyshev center property in $l^\infty(U; F)$.*

(ii) *$\mathscr{P}^N(^nE; F)$ and $\mathscr{P}_m^N(E; F)$ have the relative Chebyshev center property in $\mathscr{F}_b(E; F)$ with respect to $|\cdot|$.*

(iii) *$\mathscr{R}_{(m,n),N}^\infty(U; F)$ has the relative Chebyshev center property in $l^\infty(U; F)$ when $\dim(E) < +\infty$.*

(iv) *$\mathscr{H}_N^\infty(U; F)$ has the relative Chebyshev center property in $l^\infty(U; F)$.*

Part (i) of this theorem was proved by Roversi [12] for $\mathscr{P}^N(^nE; F)$, $n \in \mathbb{N}$, and by Deutsch *et al.* [4] for $\mathscr{P}^N(^1E; F)$ with direct proofs.

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